On the estimation of the current value of the cosmological constant

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We advance the viewpoint that, only relevant modes of the vacuum fluctuations, namely, with wavelengths conditioned by the size, homogeneity, geometry and topology of the Universe, do contribute into the cosmological constant. A formula is derived which relates the cosmological constant with the size of the Universe and the three fundamental constants: the velocity of light, Planck and Newton gravitational constants. Then the current value of the cosmological constant remarkably agrees with the value indicated by distant supernovae observations, i.e. is of the order of the critical density. Thus the cosmological constant had to be smaller than the matter density in the past and will be bigger in the future.

I. INTRODUCTION

There are observational indications for the acceleration of the expansion of the Universe [1] and that Ω_Λ attributed to the vacuum-energy density is of the order of the critical density of the Universe. Though the accuracy of observational data is still a subject for further analysis, these observations have greatly revived the interest in the long-standing problem of the cosmological constant for not only its small value, but also its closeness to the critical density of the Universe.

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The cosmological term (Λ-term) $\Lambda g_{\mu\nu}$ was introduced by Einstein to incorporate the General Relativity with the Mach principle. To interpret apparently observed behavior of redshift distribution of quasars within cosmological models of the Λ-term, Zeldovich [2] revealed the origin of the Λ-term being attributed to the zero-point energy of the vacuum: the vacuum energy-momentum tensor $\tilde{T}_{ik} = \Lambda g_{ik}$ and the negative vacuum pressure $\tilde{p}$ relating to the vacuum-energy density $\tilde{T}_{00} = \tilde{\rho} = -\tilde{p}$.

In the context of local quantum field theories in a flat space, the vacuum fluctuations of various quantum fields result in a non-vanishing vacuum-energy density given by the spectrum $\epsilon_{B,F}(|\mathbf{k}|)$ of all quantum bosonic and fermionic fields

$$\tilde{\rho} = \frac{1}{2V} \sum_{\mathbf{k}} d_{\mathbf{k}}^{B} \epsilon_B(|\mathbf{k}|) - \frac{1}{V} \sum_{\mathbf{k}} d_{\mathbf{k}}^{F} \epsilon_F(|\mathbf{k}|), \tag{1}$$

in a given volume of 3-dim space and summed over all possible $\mathbf{k}$ states with degeneracy $d_{\mathbf{k}}^{B}$ and $d_{\mathbf{k}}^{F}$. It cannot be precluded the contribution of the quantum gravity in eq.(1).

In the four dimensional flat space-time R^4 and the continuous spectrum $\epsilon_{B,F}(|\mathbf{k}|)$ of free quantum fields with a ultraviolet cutoff at the Planck scale $\Lambda_p \simeq 10^{19}$GeV, the vacuum-energy density (1) is of the order of $\Lambda_p^4 \simeq 10^{76}$GeV. This is 10^{123} times larger than the present observational data, 10^{-47}GeV, and it is unclear how they incorporate with each other unless an extremal fine-tuning is made. This is the cosmological constant problem, as discussed in details by Weinberg in [3], which challenges the fundamental theories, both the local quantum field theory and General Relativity.

The approaches to this problem can be briefly classified into three categories: (i) fundamental physics, (ii) "quintessence" and (iii) the anthropic principle. In the first category, all negative energy states are fully filled and the mean-value of the vacuum-energy density is positive, the small Λ-term is related to: the gravitational potential between virtual particles; various scales of fundamental physics, such as electroweak processes, inflationary particle creations, et al. [4]. The elegant supersymmetry forces in eq.(1) bosonic and fermionic contributions to precisely cancel each other and its breaking scale gives rise to a Λ-term. In the second category, "quintessence" [5,6] postulates a new self-interacting scalar

field Φ with a potential $V(\Phi)$, incorporating within inflationary model and quantum cosmology [7]. In the third category, the anthropic principle describes the probability of the Λ-term conditioned by the necessity for the suitable evolution of intelligent

life [8].

In this letter we present an alternative view on the problem of the cosmological constant and vacuum energy. Namely, the cosmological constant is arised only from the contributions from *the relevant vacuum fluctuations* whose wavelengths are conditioned by the *size, homogeneity, geometry* and *topology* of the Universe.

II. RELEVANT VACUUM FLUCTUATIONS

The vacuum-energy density (1), by the order of magnitude, does not strongly depend on the details of spectra $\epsilon_B(\mathbf{k})$ and $\epsilon_F(\mathbf{k})$, i.e., massive or massless, free or interacting one. Instead, it strongly depends on the number of high energy modes, since the vacuum-energy density (1) is mainly contributed from the high energy modes of vacuum fluctuations of various quantum fields at short distances.

In the general relativity, the gravity must be generated by all kinds of energy-mass, including the vacuum energy created by vacuum fluctuations of various quantum fields. However, it seems very mysterious why such a hugh vacuum-energy density $O(10^{76})$GeV (1), contributed from each spice of quantum fields, is absent in the right-handed side of the Einstein equation, thus has no any effect on the classical gravity. It is conceivable that this fact could be due to two possibilities at the distance of the Planck length L_{pl}: (i) hidden symmetries forcing the vacuum-energy density (1) to be exactly zero; (ii) a certain type of dynamical cancelations between quantum gravity's contributions and quantum field's contributions in the vacuum-energy density (1) [9].[1] We are not ambitious to cope with this

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[1]In [9] it is attempted to understand the possibility of the vacuum energy cancellation due to the energy contribution from unstable modes of quantum gravity.

problem here. Instead, we assume that the mean-value of the vacuum-energy density (1) is zero and does not contribute to the right-handed side of the Einstein equation.

On the other hand, it is equally unclear how the left-handed side of the Einstein equation describing the geometry and the large-scale Universe at present epoch can contain a non-vanishing cosmological term $\Lambda g_{\mu\nu}$. This is the problem we would like to address in this paper. We first define the notion of *the vacuum fluctuation*: it is a causally-correlated fluctuation of the vacuum, upon the zero mean-value of the vacuum-energy density (1). This has to be distinguished from the fluctuations of various quantum fields in the vacuum, which contribute to eq.(1). The non-vanishing cosmological term $\Lambda g_{\mu\nu}$ is originated from the relevant modes of the vacuum fluctuation.

Consider a complex scalar field ϕ mimicing the causally-correlated vacuum fluctuations upon the zero mean-value of the vacuum-energy density (1). The simplest coordinate-invariant action $\tilde{S}$ for the quantum scalar field $\tilde{S}$ is given by ($\hbar = c = 1$)

$$\tilde{S} = \frac{1}{2} \int d^4x \sqrt{-g} \Big[g^{\mu\nu} \phi_{,\mu} \phi^*_{,\nu} + (m^2 + \xi \mathcal{R}) \phi \phi^* \Big], \tag{2}$$

where m is an effective mass of the scalar field and ξ is the coupling constant to the Riemann scalar $\mathcal{R}$. In terms of the Riemann tensor $\mathcal{R}_{\mu\nu}$ and the energy-momentum tensor $T_{\mu\nu}$ of the classical matter, the Einstein equation is written as

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} + 8\pi G \langle \tilde{T}_{\mu\nu} \rangle_r = -8\pi G T_{\mu\nu}. \tag{3}$$

The cosmological term is described by an averaged energy-momentum tensor $\langle \tilde{T}_{\mu\nu} \rangle_r$ of the quantum scalar field ϕ:

$$\langle \tilde{T}_{\mu\nu}(x) \rangle_r = -\frac{2}{\sqrt{-g}} \frac{\delta \ell n Z_r}{\delta g^{\mu\nu}(x)}, \quad Z_r = \langle 0|0 \rangle_r = \int [\mathcal{D}\phi \mathcal{D}\phi^*]_r \exp(-\tilde{S}), \tag{4}$$

which is averaged over the relevant modes of the vacuum fluctuation with the amplitude (the partition function Z_r) of transition between relevant modes of the vacuum fluctuation in the background of Einstein equation's solution $g_{\mu\nu}(x)$ and the global topology of the Universe.

Given the action $\tilde{S}$ (2) of the quantum scalar field ϕ in the curved space-time of the Universe with its topology, we can in principle determine a unique complete and orthogonal basis of wave-functions $u_{\mathbf{k}}^r(x)$ of relevant modes $\mathbf{k}$ of the scalar field:

$$\phi(x) = \sum_{\mathbf{k}} \left(a_{\mathbf{k}} u_{\mathbf{k}}^r(x) + a_{\mathbf{k}}^\dagger u_{\mathbf{k}}^{r*}(x) \right), \tag{5}$$

where $a_{\mathbf{k}}$ is the amplitude of the $\mathbf{k}$-th relevant mode. On this relevant basis, the partition function Z_r can be computed as

$$Z_r = [\det(M^r)]^{-1}, \quad M_{\mathbf{k},\mathbf{k}'}^r = \int d^4x \sqrt{-g} u_{\mathbf{k}}^r(x)(\Delta_x + m^2 + \xi R) u_{\mathbf{k}'}^{r*}(x)$$
$$\Delta_x = \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu]. \tag{6}$$

Diagonizing the hermitian matrix M^r, we obtain

$$\ell n Z_r = - \int d^4x \sqrt{-g} \int \frac{d^4k}{(2\pi)^4} \ell n(\lambda_{\mathbf{k}}^r), \tag{7}$$

where $\lambda_{\mathbf{k}}^r$ denotes the $\mathbf{k}$-th eigen-value of the matrix M^r. Thus, the averaged energy-momentum tensor $\langle \tilde{T}_{\mu\nu} \rangle_r$ (4) is given by,

$$\langle \tilde{T}_{\mu\nu} \rangle_r = g_{\mu\nu}(x) \int \frac{d^4k}{(2\pi)^4} \ell n(\lambda_{\mathbf{k}}^r), \tag{8}$$

where we approximately neglect the functional variation $\delta g_{\mu\nu}(x)$ of eigen-values $\lambda_{\mathbf{k}}^r$ in the logarithmical function. We identify the cosmological constant,

$$\Lambda = 8\pi G \int \frac{d^4k}{(2\pi)^4} \ell n \lambda_{\mathbf{k}}^r. \tag{9}$$

This clearly indicates that the cosmological constant is determined by the eigen-values of relevant modes of the scalar field in the background of the Einstein equation.

In eq.(2), the mass of the scalar field m is scaled by the masses m_f of virtual fermions and anti-fermions that are annihilated and created in vacuum fluctuations of quantum fields. Obviously, $m \ll \Lambda_p$, otherwise there would not be any vacuum fluctuations. Analogous to eq.(1), major vacuum-fluctuation modes contributing to the averaged energy-momentum tensor (8) is stemming from the high-energy range (m, Λ_p), and we can approximately neglect

the mass term $m\phi^2$ in computations. Analogously, we approximately neglect the local interacting term $\xi\phi^2 R$, since the scale of the Riemann scalar $\mathcal{R}$ describing the large structure of the Universe is even very much smaller than m. One should not expect any significant coupling between very rapid variations of high-energy modes at short distances and the Riemann scalar $\mathcal{R}$ at long distances.

For action (2) with $m = 0$ and $\xi = 0$, the massless scalar field ϕ obeys the equation of motion,

$$M^r\phi = \Delta_x u^r_{\mathbf{k}}(x) = 0, \quad u^r_{\mathbf{k}}(x) = \mathcal{Y}^r_{\mathbf{k}}(r, \theta, \phi)\chi^r_{\mathbf{k}}(\eta). \tag{10}$$

The function $\mathcal{Y}^r_{\mathbf{k}}(r, \theta, \phi)$ fulfills the equation:

$$\Delta^3 \mathcal{Y}^r_{\mathbf{k}}(r, \theta, \phi) = -|\mathbf{k}|^2 \mathcal{Y}^r_{\mathbf{k}}(r, \theta, \phi), \quad \Delta^3 = \frac{1}{\sqrt{-h}}\partial_i[\sqrt{-h}h^{ij}\partial_j]. \tag{11}$$

The function $\chi^r_{\mathbf{k}}(\eta)$ obeys the equation

$$\frac{\partial^2\chi^r_{\mathbf{k}}(\eta)}{\partial\eta^2} + |\mathbf{k}|^2\chi^r_{\mathbf{k}}(\eta) = 0, \quad \chi^r_{\mathbf{k}}(\eta) \sim e^{i\epsilon(|\mathbf{k}|)\eta} \tag{12}$$

where $\epsilon(|\mathbf{k}|) = |\mathbf{k}|$. The solution $\chi^r_{\mathbf{k}}(\eta) \sim e^{i\epsilon(|\mathbf{k}|)\eta}$ shows the positive spectrum $\epsilon(|\mathbf{k}|)$ of the massless quantum scalar field with respect to the Killing vector ∂_η for $\eta, t \to \infty$.

III. THE CURRENT VALUE OF THE COSMOLOGICAL CONSTANT

In order to find the relevant modes in FRW Universe contributing to the cosmological constant Λ (9), we have to solve the eigenvalue equation (11) for a given size, homogeneity, geometry and topology of the Universe. We will consider for simplicity the flat FRW Universe $K = 0$, with the radius $a(t)$ and

topology $T \times R^3$ where T is the time and R^3 is the compactified spatial manifold described by the coordinates (r, θ, ϕ). The general solution of Eq.(11) can be written as

$$\mathcal{Y}^r_{\mathbf{k}}(r, \theta, \phi) \sim j_l(k_r r)Y_{lm}(\theta, \phi), \tag{13}$$

where $Y_{lm}(\theta, \phi)$ is the spherical harmonic function and $j_l(k_r r)$ is the spherical Bessel function with the radial momentum k_r and the angular quantum number

$l = 0, 1, 2....$ The eigen-value $\lambda_{\mathbf{k}}^r$ in eq.(7) is then given by,

$$(\lambda_{\mathbf{k}}^r)^2 = k_t^2 + k_r^2 + \frac{l(l+1)}{r^2}, \tag{14}$$

where k_t is the temporal component of eigen-value $\lambda_{\mathbf{k}}^r$. Integrating over k_t in eq.(9) leads to

$$\Lambda = 8\pi G \int \frac{d^3 k}{(2\pi)^3} \epsilon(|\mathbf{k}|), \quad \epsilon(|\mathbf{k}|) = \sqrt{k_r^2 + \frac{l(l+1)}{r^2}}, \tag{15}$$

up to an irrelevant integral constant independent of $|\mathbf{k}|$.

It is crucial that in our problem, due to the cosmological principle, the angular quantum number l cannot be any other values except $l = 0$ in the general solution of eq.(13). In the other words, due to the homogeneity and isotropy of the Universe, there is not a chosen point $r = 0$ of the space-time where $\langle \tilde{T}(x) \rangle_r$ and $\phi(x)$ vanish, which just requires $l = 0$. Angular quantum numbers of relevant modes of the vacuum fluctuation ϕ must be zero. In addition, the quantum scalar field ϕ is confined within the manifold of the topology R^3 and we have a simple boundary value problem with the Dirichlet condition,

$$j_\circ(\alpha_\circ^n) = 0, \quad k_r^n = \frac{\alpha_\circ^n}{a}, \tag{16}$$

where $\alpha_\circ^n$ is the n-th zero-point of the spherical Bessel function $j_\circ(x)$. The k_r^n denotes the radial momentum of relevant modes contributing the cosmological constant Λ. In our Universe $a(t) \gg 1$ and the asymptotic behavior of $j_\circ(x)$ is

$$j_\circ(k_r a(t)) \simeq \frac{2}{a(t)} \sin(k_r a), \tag{17}$$

and we find $\alpha_\circ^n = k_r^n a(t) = n\pi, n = 0, \pm 1, \pm 2, \cdots$, where $a(t)$ is the scale factor of the Universe. The positive spectrum $\epsilon(|\mathbf{k}|)$ of relevant modes of the vacuum fluctuation is given by,

$$\epsilon(k_r) = k_r = \frac{\pi n}{a}, \quad n = 0, 1, 2, \cdots. \tag{18}$$

As a result, we obtain the cosmological constant Λ,

$$\Lambda = 8\pi G \sum_l \frac{(2l+1)}{a^2(t)} \int \frac{dk_r}{(2\pi)} \sqrt{k_r^2 + \frac{l(l+1)}{a^2}} = 8\pi G \frac{\pi}{a^4} \sum_n n. \tag{19}$$

The Λ-density ρ_Λ is then given by,

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G} = \frac{1}{2} \frac{\hbar c \pi}{2a^4} N_{\max}(N_{\max}+1), \tag{20}$$

where $N_{\max}$ is the maximum number of relevant modes in the radial direction. Let us estimate the present value of the ρ_Λ. The maximum number of relevant modes in the radial direction is approximately given by

$$N_{\max} \simeq \frac{a}{L_{pl}} \simeq 10^{61}, \tag{21}$$

where the present size of the Universe $a \simeq 1 \cdot 10^{28}\, cm$ and the Planck length is $L_{pl} \simeq 1.6\, 10^{-33}\, cm$. This numerically yields the present value of the Ω_Λ

$$\rho_\Lambda = \frac{\hbar c \pi}{2a^2} L_{Pl}^{-2} \simeq 5.5\, 10^{-28}\, g\, cm^{-3}, \tag{22}$$

consistently with recent observations. The corresponding vacuum pressure is negative and thus accelerates expansion of the Universe.

The proportionality coefficient in (20) of geometry and topology of the Universe, although computations must be more complicated. An alternative approach avoiding the difficulties of the eigenvalue problem in hyperbolic manifolds is the theory of dynamical systems used for the study of the properties of CMB [10].

The probability of creation of the Universe within the framework of quantum cosmology depends not only on the matter fields but also on the cosmological constant and the topology. Particularly, the S^3 topology has the highest probability determined via the wave function of the Universe, among the considered $R \times S^3$, (K=1), $R \times H^3/\Gamma$ (K=-1) R^4 (K=0) topologies [11] and $T \times S^3$, (K=0) [12]. In the case of inflationary Universe, the probabilities for the positively and negatively curved and flat geometries become comparable.

IV. DISCUSSION

Thus, in our approach the cosmological term in the left-handed side of the Einstein equation describing the present Universe is attributed to the vacuum fluctuation ϕ whose wavelengths are conditioned by the size, homogeneity, geometry and topology of the Universe. Due to the small anisotropy of the Universe, the angular quantum numbers l of the relevant modes of the vacuum fluctuations (19) could be finite numbers. As a consequence, the cosmological constant would be angular dependent, however not more than the order of the anisotropy of the Cosmic Background radiation, 10^{-5}.

The density (20) is in fact an *absolute* value of the energy density of the vacuum fluctuations (8). If it was the energetic difference of the energy density (8) with and without presence of the classical gravity, the resulted cosmological term would be $\sim \hbar c/a^4$, which can be obtained by analogous computations leading to the Casimir effect [13].

The remarkable numerical coincidence of the obtained density with that of indicated by the supernovae observations, is prompting the idea of the variation of the Planck length by the expansion of the Universe, or

$$\frac{G}{c^4} = \text{const} \cdot a^{-2}, \tag{23}$$

which is actually in the line of Dirac's old idea of the time variation of the gravitational constant.

If this condition is fulfilled, the cosmological constant will remain constant, and its equality with the present matter density would be a chance coincidence. That is, the cosmological constant would be smaller than the matter density in the past and higher in the future. The absence of the contribution of the modes with non-zero angular quantum numbers with other formulations of the cosmological principle seems also a remarkable fact and needs further studies. The mechanism we discussed, will thus indicate the fine tuning between the macro and micro structure of the Universe, the nature of which must follow from future fundamental theories.

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